CVXPY x NASA Course 2024

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Code Generation for Quadcopter Control

Outline

Homework review

Parameters

Converting to standard form

Code generation

Solution

https://marimo.app/l/b3bjrd

This lecture

- explain the role of parameters in CVXPY
- show examples of transforming problems into standard form
- introduce CVXPYgen, a CVXPY extension for code generation
- cover more convex optimization examples

Resources

- Embedded Code Generation with CVXPY [Schaller et al.]
- `https://github.com/cvxgrp/cvxpygen`
- Differentiable Convex Optimization Layers [Agrawal et al.]
- `https://github.com/cvxgrp/cvxpylayers`

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Parameters in CVXPY

- symbolic representations of constants
- can specify sign (for use in DCP analysis)
- change value of constant without re-parsing problem

```python
# scalar parameter
alpha = cp.Parameter()

# scalar, non-negative
gamma = cp.Parameter(nonneg=True)

# matrix with shape m x n
A = cp.Parameter((m, n))
```

Parameters in CVXPY

- useful for computing trade-off curves
  - fuel use versus trajectory smoothness
  - portfolio risk versus return
  - model sparsity versus data fit
- code below updates the parameter gamma in a for loop

```
x_values = []
for val in numpy.logspace(-4, 2, 100):
    gamma.value = val
    prob.solve()
    x_values.append(x.value)
```

Parallel style trade-off curve

```python
# Use tools for parallelism in standard library.
from multiprocessing import Pool

# Function maps gamma value to optimal x.
def get_x(gamma_value):
    gamma.value = gamma_value
    result = prob.solve()
    return x.value

# Parallel computation with N processes.
pool = Pool(processes = N)
x_values = pool.map(get_x, numpy.logspace(-4, 2, 100))
```

Performance

(LASSO)

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma\|x\|_1$$

with variable $x \in \mathbf{R}^n$

- $A \in \mathbf{R}^{1000 \times 500}$, 100 values γ
- single thread time for one LASSO: 1.6 seconds (OSQP)

	for-loop	4 proc.	32 proc.	warm-start
4 core MacBook Pro	180 sec	70 sec	81 sec	31 sec
32 cores, Intel Xeon	285 sec	89 sec	31 sec	48 sec

Parametrized convex optimization

$$
\begin{array}{ll}
\text{minimize} & f_0(x,\theta) \\
\text{subject to} & f_i(x,\theta) \le 0, \quad i=1,\dots,m \\
& g_i(x,\theta) = 0, \quad i=1,\dots,p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- f_0 is the convex objective function, to be minimized
- $f_1,\dots,f_m$ are convex inequality constraint functions
- $g_1 \dots, g_p$ are affine equality constraint functions
- $\theta \in \mathbf{R}^d$ is the parameter

Exercise

`https://marimo.app/l/629s3k`

Outline

Example: Quadratic program

- original (nonnegative least squares) problem

$$\begin{array}{ll} \text{minimize} & \|Gx - h\|_2^2 \\ \text{subject to} & x \geq 0, \end{array}$$

  with variable $x \in \mathbf{R}^n$, parameters $\theta = (G, h)$

- canonicalize to form accepted by QP solver OSQP (Stellato 2020),

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} \\ \text{subject to} & l \leq A\tilde{x} \leq u \end{array}$$

  with variable $\tilde{x} \in \mathbf{R}^{\tilde{n}}$, canonical parameters $\tilde{\theta} = (P, q, A, l, u)$

Example: Canonicalization and retrieval

- canonicalize original problem using $\tilde{x} = x$ and

$$P = 2G^T G, \quad q = -2G^T h, \quad A = I, \quad l = 0, \quad u = \infty$$

- retrieve solution of original problem as $x^\star = \tilde{x}^\star$

- this example was simple and could easily be done by hand
- more complex examples much less so

Example: CVXPY code

```
import cvxpy as cp

x = cp.Variable(n, name='x')

# declare parameters
G = cp.Parameter((m, n), name='G')
h = cp.Parameter(m, name='h')

problem = cp.Problem(cp.Minimize(cp.sum_squares(G@x-h)), [x>=0])

# specify parameter values
G.value = numpy.random.randn(m, n)
h.value = numpy.random.randn(m)

problem.solve(solver='OSQP')
```

Expanding nonlinearities

- adding an L1-norm

$$\begin{array}{ll} \text{minimize} & \|Gx - h\|_2^2 + \gamma \|x\|_1 \\ \text{subject to} & x \geq 0, \end{array}$$

  with variable $x \in \mathbf{R}^n$, parameters $\theta = (G, h, \gamma)$
- how do we convert to a QP?
- the trick is introducing a new variable

Adding a new variable

- original problem

$$\begin{array}{ll} \text{minimize} & \|Gx - h\|_2^2 + \gamma \|x\|_1 \\ \text{subject to} & x \geq 0, \end{array}$$

- equivalent problem with added variable $y \in \mathbf{R}^n$

$$\begin{array}{ll} \text{minimize} & \|Gx - h\|_2^2 + \gamma \sum_{i=1}^n y_i \\ \text{subject to} & x \geq 0 \\ & -y_i \leq x_i \leq y_i, \quad i = 1, \ldots, n, \end{array}$$

- implied conditions
  - $y_i \geq 0$
  - if $x_i \geq 0$, $y_i = x_i$
  - if $x_i < 0$, $y_i = -x_i$

Exercise

`https://marimo.app/l/1eg9ov`

Solution

`https://marimo.app/l/gat8zt`

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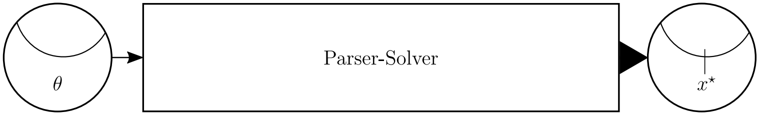
Parser-solvers

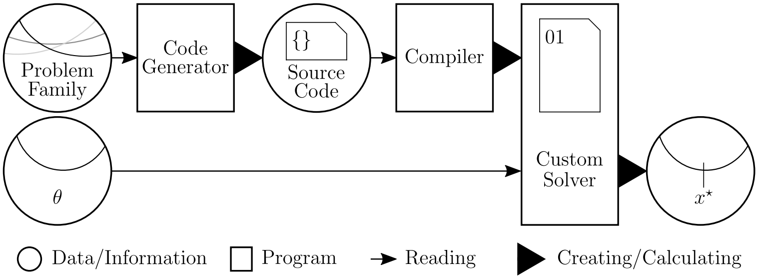
- parser-solvers canonicalize each time the problem is solved



- parser-solvers compile a *problem instance* into a *canonicalized problem instance*, then solve it
- most DSLs are parser-solvers

Code generators

- code generators compile a *problem family* into source code for a *custom solver*



- useful for
  - embedded applications, possibly with hard real-time deadlines
  - speeding up the solution of many different problem instances

CVXGEN code generator

- developed by Mattingley and Boyd in 2010
- handles problems transformable to QPs
- generates custom interior-point solver in flat, explicit C
- handles problem families with up to a few thousand parameters
- generated code suitable for real-time control systems
- used for autonomous driving, dynamic energy management, real-time trading, precision landing (*e.g.*, all SpaceX Falcon 9 and Falcon Heavy landings)

CVXGEN in action

https://blogs.nasa.gov/spacex/2019/06/25/side-boosters-have-landed/

CVXPYgen

- a new open-source code generator built on CVXPY
- developed by Schaller, Banjac, Diamond, Agrawal, Stellato, and Boyd in 2022
- generates custom canonicalizer and retrieval in flat C
- can be used with multiple solvers: OSQP, SCS (O'Donoghue 2016), ECOS (Domahidi 2013)
- first generic code generator that supports SOCPs
- supports warm-starting, which can give significant speedup
- handles high-dimensional parameters with user-defined sparsity patterns
- compiled CVXPYgen solver can be used as a custom solver for CVXPY (!)

Disciplined parametrized programming (DPP)

- restricts how parameters enter problem description, in addition to DCP rules
- for DPP-compliant problems, canonicalization and retrieval can be affine mappings (Agrawal 2019)

$$\tilde{\theta} = C \begin{bmatrix} \theta \\ 1 \end{bmatrix}, \qquad x^\star = R \begin{bmatrix} \tilde{x}^\star \\ 1 \end{bmatrix}$$

- C and R are (very) sparse matrices
- CVXPYgen generates flat C code to implement canonicalization and retrieval
- sparse-matrix-vector multiplies, using pointers or avoiding updates when possible

DPP examples

```python
x = cp.Variable()
gamma = cp.Parameter()

gamma * x # DPP
gamma**2 * x # not DPP
x / gamma # not DPP
x + gamma # DPP
gamma * x + 2 * gamma # DPP

# choose your parameters so they enter the problem
# in an affine way:
gamma2 = cp.Parameter()
gamma2.value = gamma.value ** 2
gamma2 * x # DPP
```

Example (again)

- canonicalize original nonnegative least squares problem

$$\begin{array}{ll} \text{minimize} & \|Gx - h\|_2^2 \\ \text{subject to} & x \geq 0 \end{array}$$

  to OSQP standard form

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} \\ \text{subject to} & l \leq A\tilde{x} \leq u \end{array}$$

- canonicalization shown before is not an affine mapping from θ to $\tilde{\theta}$

Example: Affine canonicalization and retrieval

- first transform to problem

$$\begin{array}{ll} \text{minimize} & \|\tilde{x}_2\|_2^2 \\ \text{subject to} & \tilde{x}_2 = G\tilde{x}_1 - h, \quad \tilde{x}_1 \geq 0, \end{array}$$

  with variable $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, $\tilde{x}_1 = x$

- canonicalization is affine:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 2I \end{bmatrix}, \quad q = 0, \quad A = \begin{bmatrix} G & -I \\ I & 0 \end{bmatrix}, \quad l = \begin{bmatrix} h \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} h \\ \infty \end{bmatrix}$$

- retrieval is affine: $x^\star = [I\ 0]\tilde{x}^\star$

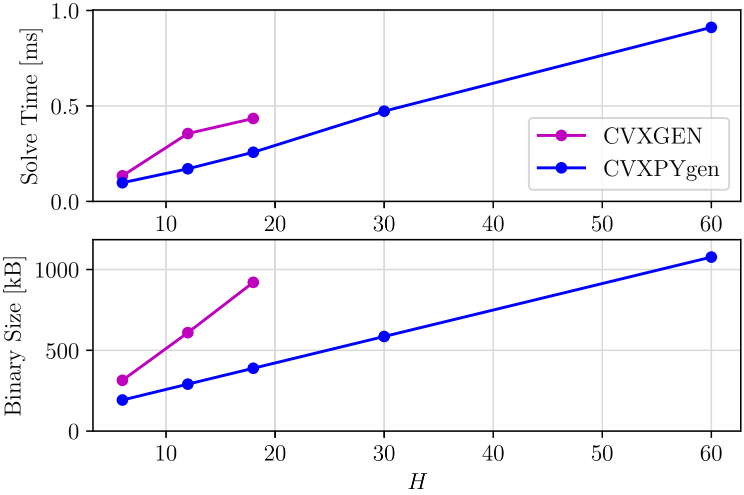
Example: CVXPY/CVXPYgen code

```python
import cvxpy as cp
from cvxpygen import cpg

# model problem
x = cp.Variable(n, name='x')
G = cp.Parameter((m, n), name='G')
h = cp.Parameter(m, name='h')
problem = cp.Problem(cp.Minimize(cp.sum_squares(G@x-h)), [x>=0])

# generate code
cpg.generate_code(problem)
```

Example: Model predictive control (MPC)

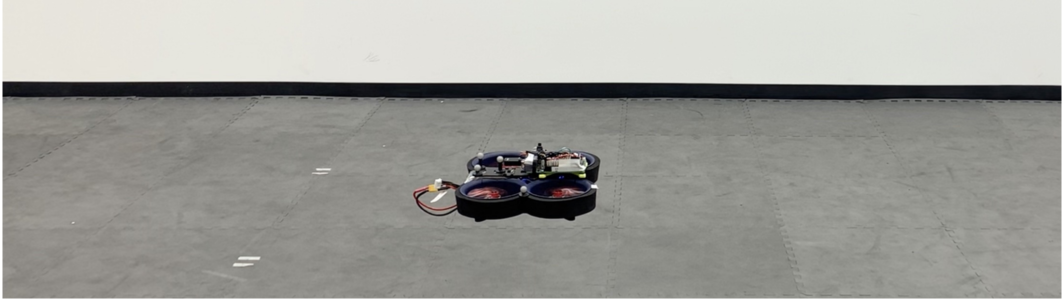
- family of MPC problems for control of a drone
- parametrized by horizon length $H \in \{6, 12, 18, 30, 60\}$
- number of variables around $10H$
- binary sizes and solve times on MacBook Pro 2.3GHz Intel i5, using gcc -O3

Comparison with CVXGEN



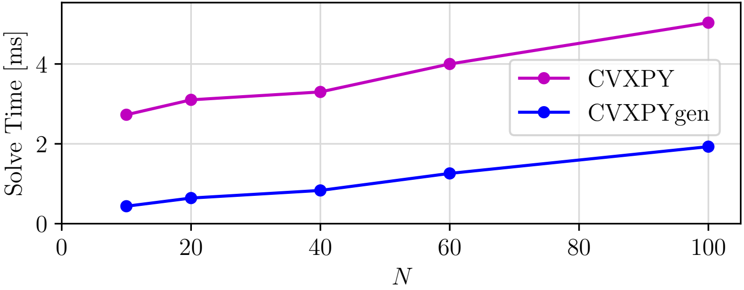
Deployment in embedded system

- use generated solver to control 14×14 cm quadcopter
- generated code compiled in a robot operating system (ROS) node
- run on drone's Intel Atom x5-Z8350 processor at 30Hz



Example: Portfolio trading

- family of portfolio optimization problems
- parametrized by number of assets $N \in \{10, 20, 40, 60, 100\}$
- number of variables around $2N$
- solve times with CVXPY and CVXPY interface to CVXPYgen



Break-even point

- *break-even point*: number of instances that need to be solved before CVXPYgen is faster than CVXPY, when we include the code generation and compilation time
- around 5000, and not too dependent on N
- typical portfolio optimization back-test involves daily trading over multiple years, with hundreds of different hyper-parameter values
- gives order 100k or more solves, well above the break-even point

Summary

CVXPYgen

- gives seemless path from prototyping in Python/CVXPY to implementation in C
- handles wider variety of problems than CVXGEN (*e.g.*, SOCPs)
- outperforms CVXGEN in terms of
  - allowable problem size
  - compiled code size
  - solve times
- gives significant speedup on general-purpose machines with many solves (compared with CVXPY)

Examples

Outline

Portfolio Optimization

Worst-Case Risk Analysis

Optimal Advertising

Portfolio allocation vector

- invest fraction w_i in asset i, $i = 1, \ldots, n$
- $w \in \mathbf{R}^n$ is *portfolio allocation vector*
- $\mathbf{1}^T w = 1$
- $w_i < 0$ means a *short position* in asset i
  (borrow shares and sell now; must replace later)
- $w \geq 0$ is a *long only* portfolio
- $\|w\|_1 = \mathbf{1}^T w_+ + \mathbf{1}^T w_-$ is *leverage*
  (many other definitions used ...)

Asset returns

- investments held for one period
- initial prices $p_i > 0$; end of period prices $p_i^+ > 0$
- asset (fractional) returns $r_i = (p_i^+ - p_i)/p_i$
- portfolio (fractional) return $R = r^T w$
- common model: r is a random variable, with mean $\mathbf{E}\, r = \mu$, covariance $\mathbf{E}(r-\mu)(r-\mu)^T = \Sigma$
- so R is a RV with $\mathbf{E}\, R = \mu^T w$, $\mathbf{var}(R) = w^T \Sigma w$
- $\mathbf{E}\, R$ is (mean) *return* of portfolio
- $\mathbf{var}(R)$ is *risk* of portfolio
  (risk also sometimes given as $\mathbf{std}(R) = \sqrt{\mathbf{var}(R)}$)

- two objectives: high return, low risk

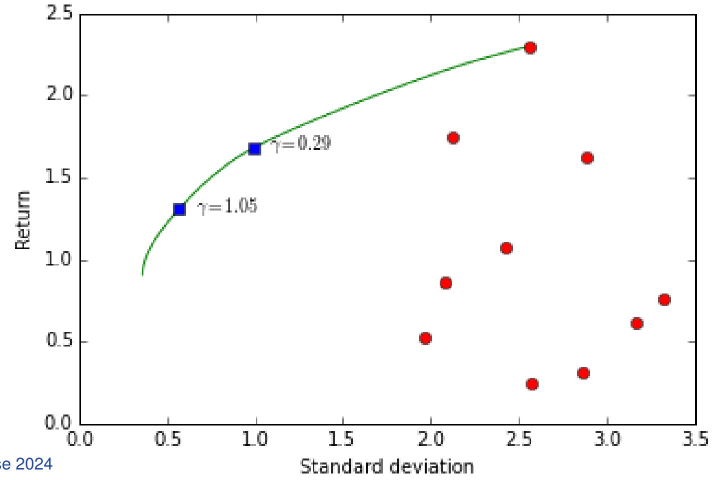
Classical (Markowitz) portfolio optimization

$$\begin{array}{ll} \text{maximize} & \mu^T w - \gamma w^T \Sigma w \\ \text{subject to} & \mathbf{1}^T w = 1, \quad w \in \mathcal{W} \end{array}$$

- variable $w \in \mathbf{R}^n$
- $\mathcal{W}$ is set of allowed portfolios
- common case: $\mathcal{W} = \mathbf{R}^n_+$ (long only portfolio)
- $\gamma > 0$ is the *risk aversion parameter*
- $\mu^T w - \gamma w^T \Sigma w$ is *risk-adjusted return*
- varying γ gives optimal *risk-return trade-off*
- can also fix return and minimize risk, etc

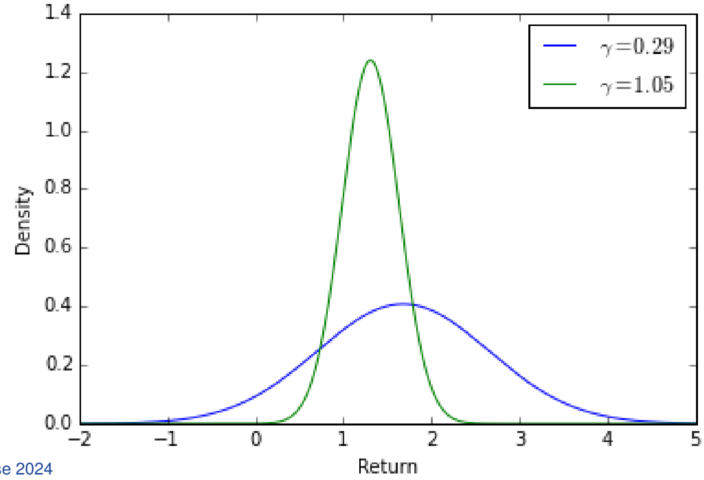
Example

optimal risk-return trade-off for 10 assets, long only portfolio



Example

return distributions for two risk aversion values



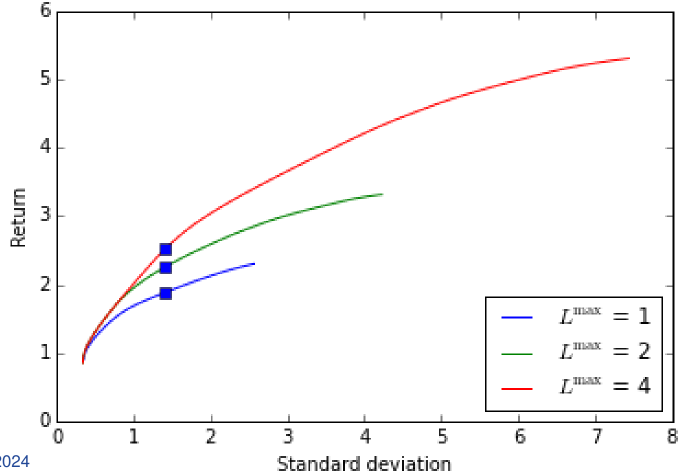
Portfolio constraints

- $\mathcal{W} = \mathbf{R}^n$ (simple analytical solution)
- leverage limit: $\|w\|_1 \leq L^{\max}$
- *market neutral*: $m^T \Sigma w = 0$
  - m_i is capitalization of asset i
  - $M = m^T r$ is *market return*
  - $m^T \Sigma w = \mathbf{cov}(M, R)$

  i.e., market neutral portfolio return is uncorrelated with market return

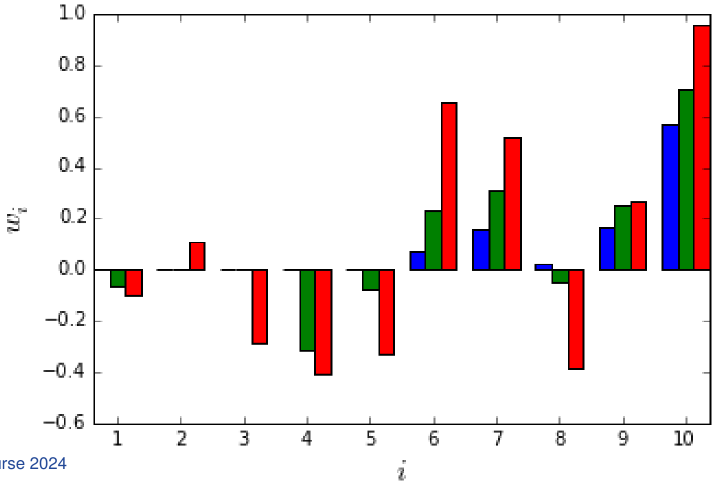
Example

optimal risk-return trade-off curves for leverage limits $1, 2, 4$



Example

three portfolios with $w^T \Sigma w = 2$, leverage limits $L = 1, 2, 4$



Variations

- require $\mu^T w \geq R^{\min}$, minimize $w^T \Sigma w$ or $\|\Sigma^{1/2} w\|_2$
- include (broker) cost of short positions,

$$s^T (w)_-, \quad s \geq 0$$

- include transaction cost (from previous portfolio w^{prev}),

$$\kappa^T |w - w^{\text{prev}}|^\eta, \quad \kappa \geq 0$$

common models: $\eta = 1,\ 3/2,\ 2$

Factor covariance model

$$\Sigma = F\tilde{\Sigma}F^T + D$$

- $F \in \mathbf{R}^{n \times k}$, $k \ll n$ is *factor loading matrix*
- k is number of factors (or sectors), typically 10s
- F_{ij} is loading of asset i to factor j
- D is diagonal matrix; $D_{ii} > 0$ is *idiosyncratic risk*
- $\tilde{\Sigma} > 0$ is the *factor covariance matrix*

- $F^T w \in \mathbf{R}^k$ gives portfolio *factor exposures*
- portfolio is *factor j neutral* if $(F^T w)_j = 0$

Portfolio optimization with factor covariance model

$$
\begin{array}{ll}
\text{maximize} & \mu^T w - \gamma \left(f^T \tilde{\Sigma} f + w^T D w \right) \\
\text{subject to} & \mathbf{1}^T w = 1, \quad f = F^T w \\
& w \in \mathcal{W}, \quad f \in \mathcal{F}
\end{array}
$$

- variables $w \in \mathbf{R}^n$ (allocations), $f \in \mathbf{R}^k$ (factor exposures)
- $\mathcal{F}$ gives factor exposure constraints

- computational advantage: $O(nk^2)$ vs. $O(n^3)$

Example

- 50 factors, 3000 assets
- leverage limit $= 2$
- solve with covariance given as
  - single matrix
  - factor model
- CVXPY/OSQP single thread time

covariance	solve time
single matrix	173.30 sec
factor model	0.85 sec

Outline

Portfolio Optimization

Worst-Case Risk Analysis

Optimal Advertising

Covariance uncertainty

- single period Markowitz portfolio allocation problem
- we have fixed portfolio allocation $w \in \mathbf{R}^n$
- return covariance Σ not known, but we believe $\Sigma \in \mathcal{S}$
- $\mathcal{S}$ is convex set of possible covariance matrices
- risk is $w^T \Sigma w$, a *linear function of* Σ

Worst-case risk analysis

- what is the worst (maximum) risk, over all possible covariance matrices?
- worst-case risk analysis problem:

$$\begin{array}{ll} \text{maximize} & w^T \Sigma w \\ \text{subject to} & \Sigma \in \mathcal{S}, \quad \Sigma \succeq 0 \end{array}$$

  with variable Σ
- ...a convex problem with variable Σ

- if the worst-case risk is not too bad, you can worry less
- if not, you'll confront your worst nightmare

Example

- $w = (-0.01, 0.13, 0.18, 0.88, -0.18)$
- optimized for Σ^{nom}, return 0.1, leverage limit 2
- $\mathcal{S} = \left\{\Sigma^{\text{nom}} + \Delta \ : \ |\Delta_{ii}| = 0,\ |\Delta_{ij}| \leq 0.2\right\}$,

$$\Sigma^{\text{nom}} = \begin{bmatrix} 0.58 & 0.2 & 0.57 & -0.02 & 0.43 \\ 0.2 & 0.36 & 0.24 & 0 & 0.38 \\ 0.57 & 0.24 & 0.57 & -0.01 & 0.47 \\ -0.02 & 0 & -0.01 & 0.05 & 0.08 \\ 0.43 & 0.38 & 0.47 & 0.08 & 0.92 \end{bmatrix}$$

Example

- nominal risk $= 0.168$
- worst case risk $= 0.422$

$$\text{worst case } \Delta = \begin{bmatrix} 0 & 0.04 & -0.2 & -0. & 0.2 \\ 0.04 & 0 & 0.2 & 0.09 & -0.2 \\ -0.2 & 0.2 & 0 & 0.12 & -0.2 \\ -0. & 0.09 & 0.12 & 0 & -0.18 \\ 0.2 & -0.2 & -0.2 & -0.18 & 0 \end{bmatrix}$$

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Worst-Case Risk Analysis

Optimal Advertising

Ad display

- m advertisers/ads, $i = 1, \ldots, m$
- n time slots, $t = 1, \ldots, n$
- T_t is total traffic in time slot t
- $D_{it} \geq 0$ is number of ad i displayed in period t
- $\sum_i D_{it} \leq T_t$
- contracted minimum total displays: $\sum_t D_{it} \geq c_i$
- goal: choose D_{it}

Clicks and revenue

- C_{it} is number of clicks on ad i in period t
- click model: $C_{it} = P_{it}D_{it}$, $P_{it} \in [0, 1]$
- payment: $R_i > 0$ per click for ad i, up to budget B_i
- ad revenue

$$S_i = \min \left\{ R_i \sum_t C_{it}, B_i \right\}$$

…a concave function of D

Ad optimization

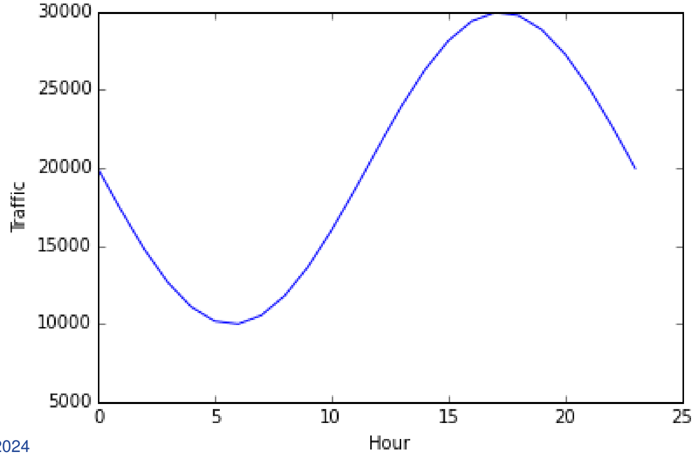
- choose displays to maximize revenue:

$$\begin{array}{ll} \text{maximize} & \sum_i S_i \\ \text{subject to} & D \geq 0, \quad D^T \mathbf{1} \leq T, \quad D\mathbf{1} \geq c \end{array}$$

- variable is $D \in \mathbf{R}^{m \times n}$
- data are T, c, R, B, P

Example

- 24 hourly periods, 5 ads (A through E)
- total traffic:



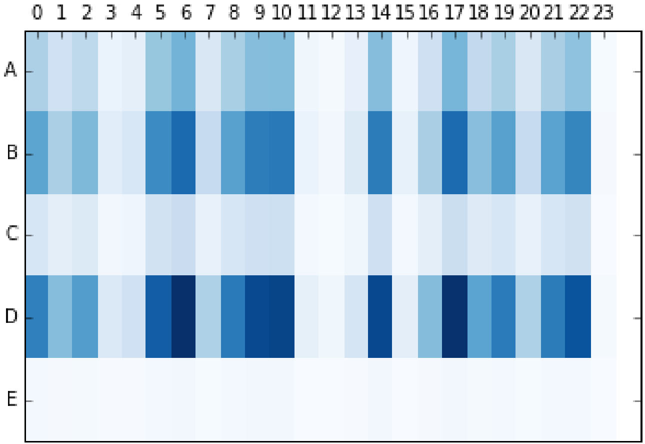
Example

- ad data:

Ad	A	B	C	D	E
c_i	61000	80000	61000	23000	64000
R_i	0.15	1.18	0.57	2.08	2.43
B_i	25000	12000	12000	11000	17000

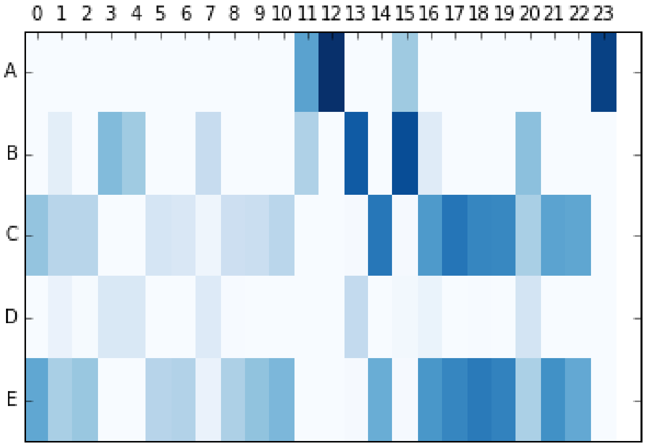
Example

P_{it}



Example

optimal D_{it}



Example

ad revenue

Ad	A	B	C	D	E
c_i	61000	80000	61000	23000	64000
R_i	0.15	1.18	0.57	2.08	2.43
B_i	25000	12000	12000	11000	17000
$\sum_t D_{it}$	61000	80000	148116	23000	167323
S_i	182	12000	12000	11000	7760